ISRAEL JOURNAL OF MATHEMATICS 96 (1996), 399–418

DISCRETE GROUPS OF SLOW SUBGROUP GROWTH*

BY

ALEXANDER LUBOTZKY

Institute of Mathematics, The Hebrew University of Jerusalem
Jerusalem 91904, Israel
e-mail: alexlub@math.huji.ac.il

AND

LÁSZLÓ PYBER

Mathematical Institute, Hungarian Academy of Science
P.O.B. 127 Budapest, Hungary H-1364
e-mail: pyber@novell.math-inst.hu

AND

ANER SHALEV

Institute of Mathematics, The Hebrew University of Jerusalem
Jerusalem 91904, Israel
e-mail: shalev@math.huji.ac.il

ABSTRACT

It is known that the subgroup growth of finitely generated linear groups is either polynomial or at least $n^{\frac{\log n}{\log\log n}}$. In this paper we prove the existence of a finitely generated group whose subgroup growth is of type $n^{\frac{\log n}{(\log\log n)^2}}$. This is the slowest non-polynomial subgroup growth obtained so far for finitely generated groups. The subgroup growth type $n^{\log n}$ is also realized. The proofs involve analysis of the subgroup structure of finite alternating groups and finite simple groups in general. For example, we show there is an absolute constant c such that, if T is any finite simple group, then T has at most $n^{c\log n}$ subgroups of index n.

* The first and third authors acknowledge support of The Israel Science Foundation, administered by The Israel Academy of Sciences and Humanities. The second author acknowledges support of the Hungarian National Foundation for Scientific Research, Grant No. T7441.
Received January 26, 1995

1. Introduction

Let G be a finitely generated residually finite group. For $n \geq 1$, let $a_n(G)$ denote the number of subgroups of index n in G. The series $\{a_n(G)\}$ has been the subject of intense investigation in the past decade (see [L2], [L3] and the references therein). Given a function f, we say that G has **subgroup growth type** (or simply **growth type**) f, if there are positive constants b, c such that

$$a_n(G) \leq f(n)^c \quad \text{for all } n,$$

and

$$a_n(G) \geq f(n)^b \quad \text{for infinitely many } n.$$

Groups of polynomial subgroup growth (also referred to as PSG groups) have growth type n (provided they are not almost cyclic). These groups are virtually soluble of finite rank [LMS]. As for non-polynomial growth types, the slowest one which has been realized is $n^{\frac{\log n}{\log\log n}}$. For example, this is the growth type of $\mathrm{SL}_d(\mathbb{Z})$ ($d \geq 3$) and of other arithmetic groups in characteristic 0 which satisfy the congruence subgroup property [L1]. In fact, it is shown in [L1] that $n^{\frac{\log n}{\log\log n}}$ is the minimal growth type of linear non-PSG groups. Moreover, the growth type of soluble linear non-PSG groups is at least 2^{n^ϵ} [SSh].

The main purpose of this paper is to realize a non-polynomial growth type which out-does the bound for linear groups. Indeed we have:

THEOREM 1.1: *There exists a finitely generated group whose subgroup growth type is* $n^{\frac{\log n}{(\log\log n)^2}}$.

We conjecture that this result is best possible, in the sense that the growth type of any finitely generated non-PSG (discrete) group is at least $n^{\frac{\log n}{(\log\log n)^2}}$. This seems to be quite a challenging problem. In fact at the moment it is not even known whether there is any gap between polynomial and non-polynomial growth for finitely generated groups.

Note, however, that in non-finitely generated groups — or in finitely generated profinite groups — arbitrarily slow non-polynomial growth types can be realized [Sh2]. On the other hand, for pro-p groups there is a gap between polynomial and non-polynomial growth, and the minimal non-polynomial growth type is $n^{\log n}$ [Sh1].

The gap question is a particular case of a more general problem, namely, finding out which growth types can be realized using finitely generated groups. While it seems plausible that $\mathrm{SL}_d(\mathbb{F}_p[t])$ ($d \geq 3$) and many other arithmetic groups in positive characteristic have growth type $n^{\log n}$, this has not yet been verified (it is shown in [L1] that the growth type of such groups is bounded below by $n^{\log n}$ and bounded above by $n^{\log^2 n}$). In fact, while there are many pro-p groups whose growth type is $n^{\log n}$ (cf. [Sh1], [LSh]), this growth type has not yet been realized for finitely generated (discrete) groups. Using constructions similar to those used in the proof of Theorem 1.1 we obtain the following.

THEOREM 1.2: *There exists a finitely generated group whose subgroup growth type is* $n^{\log n}$.

The groups obtained in Theorems 1.1 and 1.2 are both not linear. Their profinite completion is a direct product of a procyclic group with a cartesian product of infinitely many (pairwise non-isomorphic) finite simple groups T_i. In Theorem 1.1 the groups T_i are alternating, and the proof eventually boils down to counting subgroups in finite alternating groups A_t.

We prove the following.

THEOREM 1.3: *There exists a constant c such that*

$$a_n(A_t) \leq n^{\frac{c \log n}{(\log \log n)^2}}$$

for all t, n.

In Theorem 1.2, the groups T_i are classical groups of Lie type (of unbounded rank), and in the course of the proof we determine their growth behaviour. In fact we obtain the following.

THEOREM 1.4: *There exists a constant c such that, for any finite simple group* T, *we have*

$$a_n(T) \leq n^{c \log n}$$

for all n.

We also obtain the following information on the number of generators of subgroups of simple groups.

PROPOSITION 1.5: *There exists a constant c such that, for any finite simple group* T *and for any subgroup* $H < T$, *we have*

$$d(H) \leq c \log |T\colon H|.$$

We also show that the upper bounds in 1.3, 1.4 and 1.5 are best possible (apart from the values of the constants). These results are of independent interest and could probably be used in other contexts.

Our final result deals with subnormal subgroups of finitely generated groups. Let $a_n^{\lhd\lhd}(G)$ denote the number of subnormal subgroups of index n in the group G. If $a_n^{\lhd\lhd}(G) \leq n^c$ for some c and for all n, we say that G has **polynomial subnormal subgroup growth.** The problem of characterizing finitely generated groups of polynomial subnormal subgroup growth is an interesting one. In fact several results which were obtained in the context of subgroup growth are really about the growth of subnormal subgroups (see, e.g., [MS, Theorem 3.9] showing that prosoluble groups of polynomial subnormal subgroup growth have finite rank, as well as the main results of [SSh]). One could therefore hope that finitely generated groups of polynomial subnormal subgroup growth have a nice structure (e.g. that they are virtually soluble, or linear). We show below that this is not the case. In fact it turns out that the groups constructed in Theorems 1.1 and 1.2 have polynomial subnormal subgroup growth. We therefore obtain:

COROLLARY 1.6: *Finitely generated residually finite groups of polynomial subnormal subgroup growth need not be virtually soluble, or linear.*

Additional results on subnormal subgroups in finitely generated groups are included in the forthcoming paper [LPSh]. We note that ideas from the theory of permutation groups are also useful in the study of groups of very fast subgroup growth (see [PSh]).

Some words on the structure of this paper. In Section 2 we study the growth behaviour of finite alternating groups and prove Theorem 1.3. Section 3 is devoted to the proof of Theorem 1.4 and Proposition 1.5. The proofs of 1.3–1.5 rely (among other things) on small index theorems obtained by Liebeck in [Li].

In Section 4 we pass from finite groups to infinite groups. We define the groups used in Theorems 1.1 and 1.2 and determine their profinite completions. In fact the group used in Theorem 1.1 is a variant of a group which was originally constructed by B. H. Neumann [N] and reused in [LW] for different purposes. Replacing the symmetric groups in Neumann's construction by groups of Lie type, we obtain the group occurring in Theorem 1.2 (originally constructed in [LW]). It should be mentioned that, though these groups have already appeared before, computing their profinite completion is not entirely obvious.

Finally, in Section 5 we put everything together and prove Theorem 1.1, Theorem 1.2, and Corollary 1.6.

Our notation is rather standard. The minimal number of generators of a group G is denoted by $d(G)$. The rank $r(G)$ of G is the minimal integer r such that all finitely generated subgroups of G are r-generated. For profinite groups these notions are interpreted topologically. Similarly, when counting finite index subgroups of a profinite group G we restrict ourselves to open subgroups. Let $s_n(G)$ denote the number of subgroups of index at most n in a group G. The total number of subgroups of G is denoted by $s(G)$. The profinite completion of an abstract group Γ is denoted by $\widehat{\Gamma}$. The closure of an abstract subgroup L of a profinite group G is denoted by $\overline{L}$. All logarithms are to the base 2.

2. Growth of alternating groups

We start with the following preliminary result.

LEMMA 2.1: *Let A be a subgroup of* $\mathrm{Sym}(\Delta)$ *with orbits* $\Delta_1, \ldots, \Delta_r$. *Suppose that every orbit* Δ_i *has a partition into blocks of imprimitivity* Δ_i^j *such that*

(1) $|\Delta_i^j| \geq b$ *for some fixed* $b \geq 5$.

(2) *The setwise stabiliser* A_i^j *of* Δ_i^j *acts on* Δ_i^j *as* $\mathrm{Sym}(\Delta_i^j)$ *or* $\mathrm{Alt}(\Delta_i^j)$.

Then every normal subgroup N of A can be generated by $3|\Delta|/b$ *elements.*

Proof: Denote by K the kernel of the action of A on the set of all blocks Δ_i^j. Then A/K has an embedding into S_t where $t \leq |\Delta|/b$ is the total number of blocks.

Now, K is a normal subgroup of A_i^j for each i, j, so K acts on Δ_i^j as $\mathrm{Sym}(\Delta_i^j)$, $\mathrm{Alt}(\Delta_i^j)$, or 1. This gives a natural embedding of K into a direct product of symmetric groups. Denote by H the product of the corresponding alternating groups. Then $K_0 = H \cap K$ is clearly a subdirect product of alternating groups and so it is a direct product of at most t alternating groups. Furthermore, K/K_0 is an elementary abelian 2-group of rank at most t.

Suppose now that N is a normal subgroup of A. Then $N/N \cap K \cong NK/K \leq A/K \leq S_t$, and therefore $N/N \cap K$ can be generated by at most $t-1$ elements [Jer]. Similarly, $N \cap K/N \cap K_0$ can be generated by at most t elements (since it is embedded in K/K_0).

It is easy to see that $N \cap K_0$ (as a normal subgroup of K_0) is a direct product of at most t alternating groups (of degree ≥ 5) and therefore it can be generated by $t+1$ elements [Wi]. The result follows. ∎

PROPOSITION 2.2: *There exists an absolute constant* c *such that*

$$a_n(A_t) \le n \cdot 2^{c(\log n/\log t)^2} \quad \textit{for all } n.$$

Proof: It suffices to prove a similar bound on $a_n(S_t)$ (as $a_n(A_t) \le a_{2n}(S_t)$). Let $G \le \mathrm{Sym}(\Omega)$ be a subgroup of index n, where $|\Omega| = t$. We shall assume (as we may) that t is rather large.

CASE 1: Suppose that $n \le 2^{t^{4/5}}$.

By a result of Liebeck [Li, Lemma 1.1] there is an integer $m < t/2$ and a subgroup H_m of A_m such that

$$H = H_m \times A_{t-m} \le G \le S_m \times S_{t-m},$$

and $|G\colon H| \le 2$.

Clearly we have $n \ge \binom{t}{m} \ge 2^m$ and therefore $m \le t^{4/5}$ (by our assumption on n). It follows that $\binom{t}{m} \ge t^{m/5}$ and therefore $m \le 5\frac{\log n}{\log t}$.

It is proved in [Py] (using elementary arguments) that the number of subgroups of S_m is at most 2^{am^2} for some fixed constant $a > 0$. The number of choices for m is at most $5\log n/\log t$. The number of choices for an m element subset of Ω is $\binom{t}{m} \le n$. The group $H_m \le S_m$ can be chosen in at most $2^{am^2} \le 2^{25a(\log n/\log t)^2}$ ways; once $H = H_m \times A_{t-m}$ is given, G is obtained by adding to it (the inverse image of) a single (possibly trivial) element from $S_m \times S_{t-m}/A_{t-m}$, and there are at most $2m! \le 2^{m^2} \le 2^{25(\log n/\log t)^2}$ ways to choose it. Altogether we conclude that the number of choices for G in this case is at most

$$n \cdot \frac{5\log n}{\log t} 2^{25(a+1)(\frac{\log n}{\log t})^2} \le n \cdot 2^{c(\frac{\log n}{\log t})^2},$$

where c is some absolute constant. This implies the required conclusion.

CASE 2: Suppose that $n > 2^{t^{4/5}}$.

Set $b = \sqrt{t}$. Let Δ be the largest G-invariant subset of Ω such that G acts on Δ as a permutation group $A \le \mathrm{Sym}(\Delta)$ satisfying the conditions of Lemma 2.1. Then every normal subgroup of A can be generated by at most $3\sqrt{t}$ elements.

CLAIM: $|\Omega \setminus \Delta| \le \frac{2\log n}{\log t - 7}$.

To show this, denote the orbits of G on $\Omega \setminus \Delta$ by $\Omega_1, \ldots, \Omega_k$. Let $m_i = |\Omega_i|$ and $m = |\Omega \setminus \Delta|$. Let $G_i \le \mathrm{Sym}(\Omega_i)$ be the permutation group which G induces on Ω_i $(1 \le i \le k)$. Let Ω_i^j be a partition of Ω_i into blocks of imprimitivity for G_i, and let G_i^j be the permutation group induced on Ω_i^j by the setwise stabiliser of Ω_i^j in G. Then G_i^j is a primitive subgroup of $\mathrm{Sym}(\Omega_i^j)$, and for fixed i the permutation groups G_i^j are all equivalent.

Fix i and suppose first that (for some j, equivalently for all j) G_i^j does not contain $\mathrm{Alt}(\Omega_i^j)$. By a well known result of Praeger and Saxl [PS], the order of a primitive permutation group of degree l not containing A_l cannot exceed 4^l. It follows that $|G_i^j| \le 4^{|\Omega_i^j|}$. Therefore

$$|G_i| \le 4^{m_i}(m_i/2)! \le 4^{m_i}(t/2)^{m_i/2} = (8t)^{m_i/2}.$$

Suppose now that G_i^j does contain $\mathrm{Alt}(\Omega_i^j)$ (for all j). Then by the choice of Δ, Ω_i has a partition into blocks Ω_i^j of size $2 \le b_i \le \sqrt{t}$. We have

$$|G_i| \le (b_i!)^{m_i/b_i}(m_i/b_i)! \le ((b_i)!m_i/b_i)^{m_i/b_i} = ((b_i-1)!m_i)^{m_i/b_i}.$$

Since $b_i \le \sqrt{t}$ and $m_i \le t$, it follows that $(b_i-1)!m_i \le t^{b_i/2}$ (recall that t is large). Therefore $|G_i| \le (t^{b_i/2})^{m_i/b_i} = t^{m_i/2}$ in this case.

We see that, in any case, we have $|G_i| \le (8t)^{m_i/2}$.

Let $B = G^{\Omega \setminus \Delta}$, the permutation group which G induces on $\Omega \setminus \Delta$. The above discussion shows that

$$|B| \le \prod_i |G_i| \le \prod_i (8t)^{m_i/2} = (8t)^{m/2}.$$

Since $|G| \le |B||S_{t-m}| \le (8t)^{m/2}(t-m)!$ we obtain that

$$n = |S_t\colon G| \ge \frac{t!}{(t-m)!(8t)^{m/2}} \ge \frac{(t/4)^m}{(8t)^{m/2}} = \left(\frac{t}{2^7}\right)^{m/2},$$

proving the claim.

Note that, with the above notation, G is a subdirect product of $A \le S_{t-m}$ and $B \le S_m$. Therefore G contains the group $N = (G \cap A) \times (G \cap B)$, $G \cap A$ is a normal subgroup of A and $G/N \cong A/G \cap A$.

Now, there are at most $t \le n$ ways to choose $|\Delta|$, and given $|\Delta| = t - m$ there are at most $\binom{t}{m} \le n$ ways to choose Δ (recall that $G \le S_m \times S_{t-m}$). We see that the number of choices for Δ is at most n^2. Using the above claim we conclude that the number of choices for $G \cap B$ is at most $2^{a(2\log n/(\log t - 7))^2}$. By Lemma 2.1, both $G \cap A$ and G/N (being isomorphic to a quotient of A) can be generated by $3\sqrt{t}$ elements. In particular we have at most $(t!)^{3\sqrt{t}}$ choices for $G\cap A$. Assuming $G\cap A, G\cap B$ are both given, N is determined, and so G can be chosen in at most $(t!)^{3\sqrt{t}}$ ways. Note that, using our assumption that $n \geq 2^{t^{4/5}}$, we obtain

$$(t!)^{6\sqrt{t}} \leq 2^{6t^{3/2}\log t} \leq 2^{(\log n/\log t)^2},$$

for large enough t. It follows that the number of choices for G given $G\cap B$ cannot exceed $2^{(\log n/\log t)^2}$. Putting everything together we see that G can be chosen in at most

$$n^2 \cdot 2^{a(2\log n/(\log t - 7))^2} \cdot 2^{(\log n/\log t)^2}$$

ways. Note that, by our assumption on n, this expression is bounded above by $2^{c(\log n/\log t)^2}$ (for a suitable c). This yields the desired conclusion.

Proposition 2.2 is proved. ∎

By considering subgroups of the form $H_m \times A_{t-m}$ where H_m ranges over all subgroups of rank $m/4$ of a given elementary abelian 2-subgroup of rank $m/2$ in S_m, we see that Proposition 2.2 is best possible, apart from the value of the constant c; we also see that we must have $c \geq 1/16$. The above argument can be used to obtain an estimate like $c \leq 100$. This can be drastically improved using the Classification Theorem. However, to obtain a sharp estimate for c (when t is large) seems to be a rather difficult problem.

Proof of Theorem 1.3: We may assume $n \leq t!$ (otherwise $a_n(A_t) = 0$). Therefore $n \leq 2^{t^2}$, which implies that $\log\log n \leq 2\log t$. This yields

$$2^{c(\log n/\log t)^2} \leq 2^{4c(\log n/\log\log n)^2} = n^{4c\log n/(\log\log n)^2}.$$

Therefore Theorem 1.3 follows from Proposition 2.2. ∎

It is clear from the previous remark that the bound in Theorem 1.3 is best possible (apart from the value of the constant).

3. Growth of simple groups of Lie type

We require the following description of subgroups of simple groups of Lie type. Let T be a simple group of Lie type and let $T_0 \leq T$ be a quasisimple subgroup of Lie type. We say that T_0 is a **naturally embedded subgroup** of T if the natural module of T_0 is a subspace of the natural module of T. In particular, T and T_0 have the same characteristic. We denote the covering group of T_0 by $\widetilde{T_0}$.

PROPOSITION 3.1: *There exists an absolute constant c with the following property: if T is a finite simple group of Lie type, and H is any subgroup of T, then there is a subgroup $T_0 \leq T$ satisfying*

(i) $T_0 = \widetilde{T}_0$ *is quasisimple and naturally embedded in* T, *or* $T_0 = 1$.

(ii) $T_0 \leq H$.

(iii) $|T: T_0| \leq |T: H|^c$.

Proof: It is well known that, if T is a simple group of Lie type of Lie rank k having a subgroup of index $n > 1$, then $|T| \leq n^c$ where c is a constant depending on k (see [KlLi, 5.2.2] and [LaSe]). Hence, if k is bounded, then the conclusion holds with $T_0 = 1$.

It remains to deal with groups of large Lie rank. In particular we may assume that T is a classical group. We can also assume that the index of H is quite small, i.e. $|T: H| < |T|^{1/c}$ (where c is a fixed large constant).

The existence of the quasisimple subgroup T_0 now follows from the detailed descriptions of small index subgroups of classical groups, given in [Li, Section 5] for $\mathrm{SL}_n(q)$, and in [Li, Section 6] for other classical groups (in fact, not all the classical groups are discussed in [Li], but they can all be treated in a similar manner). ∎

For the purpose of proving Theorem 1.2, we only need the case $T = \mathrm{PSL}_n(q)$. For completeness we analyse this case below, without relying on the results of [Li], which are quite technical. It will follow from our analysis that $c = 4$ will do in this case, and that we have $T_0/Z(T_0) \cong \mathrm{PSL}_k(q)$ for a suitable k. For simplicity we shall assume below that $q > 3$.

PROPOSITION 3.2: *Let k, n be integers such that $1 \leq n/2 < k \leq n$. Let V be an n-dimensional linear space over $\mathbb{F}_q$ ($q > 3$) and let $G = \mathrm{SL}(V)$. Let $H \leq G$ be a subgroup satisfying*

$$|H| \geq q^{n^2-1-(k-1)(n-k+1)}.$$

Then there exists a k-dimensional subspace $U \leq V$ such that

$$H \geq \mathrm{SL}(U).$$

Proof: We use induction on n. For $n = 2$ the result follows immediately from the fact that $\mathrm{SL}_2(q)$ has no proper subgroups of index less than q. So let $n \geq 3$. Note that

$$|G: H| < q^{(k-1)(n-k+1)} \leq q^{n^2/4} \leq q^{n(n-1)/2}.$$

It follows from a theorem of Kantor [K, Theorem 1] that either (a) $H = G$, or (b) $n = 2m$ and $H = \mathrm{Sp}(V)$, or (c) H is a reducible subgroup. It is easy to rule out case (b) by order considerations, since we have $n \geq 4$ and $|H| < q^{n^2/2+n/2} < q^{n^2-1-n^2/4}$ in this case. It remains to consider the case where H is reducible.

Let $M < G$ be a maximal parabolic subgroup containing H. Then M is the stabilizer of a subspace W of V, and $M \geq \mathrm{SL}(V_0)$, where $V_0 \leq V$ is either W or a complement of W, and $m = \dim V_0 \geq n/2$. It is also clear that $m < n$. We have $|H| \leq |M| < q^{n^2-1-m(n-m)}$, and this implies $m \geq k$ by the assumption on $|H|$. Let $G_0 = \mathrm{SL}(V_0)$ and $H_0 = G_0 \cap H$. Since $|M: G_0| \leq q^{n(n-m)}$ and $H \leq M$ we see that

$$|H_0| \geq |H| q^{-n(n-m)} \geq q^t,$$

where $t = n^2-1-(k-1)(n-k+1)-n(n-m) = m^2+m(n-m)-1-(k-1)(n-k+1) \geq m^2 - 1 + (k-1)(n-m) - (k-1)(n-k+1) = m^2 - 1 - (k-1)(m-k+1)$. Replacing V by V_0 and $H \leq G$ by $H_0 \leq G_0$ and using the induction hypothesis we conclude that $H_0 \geq \mathrm{SL}(U)$ for some k-dimensional subspace $U \leq V_0$. Thus $H \geq \mathrm{SL}(U)$ as required. ∎

We note that [K, Theorem 1] also deals with the cases $q = 2, 3$. Combining that result with the above arguments it is easy to obtain a suitable modification of Proposition 3.2 for these cases.

COROLLARY 3.3: *Let $G = \mathrm{SL}_n(q)$ $(q > 3)$ and let $H \leq G$ be a subgroup of index at most $q^{(n^2-1)/4}$. Then G has a naturally embedded subgroup $G_0 \cong \mathrm{SL}_k(q)$ for some k, such that $G_0 \leq H$ and*

$$|G: G_0| \leq |G: H|^3.$$

Proof: If $n = 2$ then we must have $H = G$. We can therefore assume that $n \geq 3$. The assumption on $|G: H|$ implies that there exists an integer $k > n/2$ with the property that $|G: H| \leq q^{(k-1)(n-k+1)}$. Suppose k is maximal with this property. By Proposition 3.2 there is a subgroup $G_0 \leq H$, $G_0 \cong \mathrm{SL}_k(q)$, naturally embedded in G. We may assume that $k < n$. It is easy to see that $|G: G_0| \leq q^{n^2-k^2+1}$. Since $k \geq (n+1)/2$ we have

$$3k(n-k) - (n^2 - k^2 + 1) = (2k - n)(n - k) - 1 \geq 0.$$

Note that $|G: H| > q^{k(n-k)}$ by the choice of k. Putting everything together we obtain

$$|G: G_0| \leq q^{n^2-k^2+1} \leq q^{3k(n-k)} \leq |G: H|^3,$$

as required. ∎

Again, it is easy to obtain a slightly weaker result for $q = 2, 3$.

Proof of Proposition 1.5: It suffices to consider groups of Lie type and alternating groups. Suppose T is a simple group of Lie type, $H < T$, and let $T_0 \leq H$ and c be as in Proposition 3.1. Then $d(T_0) \leq 2$ and $|T\colon T_0| \leq |T\colon H|^c$. It is clear that

$$d(H) \leq d(T_0) + \log|H\colon T_0| \leq 2 + \log|T\colon H|^{c-1} = (c-1)\log|T\colon H| + 2.$$

The result follows in this case.

Now let $T = A_t$ be an alternating group. We can assume that t is large (say $t \geq 24$). Let $H \leq T$. We shall show that $d(H) \leq \log|T\colon H|$.

Case 1: $|T\colon H| \geq 2^{t/2}$. Note that $r(A_t) \leq t/2$ by a result of A. McIver and P. M. Neumann [MN]. Hence $d(H) \leq t/2 \leq \log|T\colon H|$.

Case 2: $|T\colon H| < 2^{t/2}$. Then, by a previously quoted result from [Li], there is an integer $m < t/2$ and a subgroup $H_m \leq S_m$ such that

$$H_m \times A_{t-m} \leq H \leq S_m \times S_{t-m},$$

and $|H\colon H_m \times A_{t-m}| \leq 2$. Note that $d(H_m) \leq m/2$ by [MN]. It follows that

$$d(H) \leq d(H_m) + d(A_{t-m}) + 1 \leq m/2 + 3.$$

Now, we have $|T\colon H| \geq \binom{t}{m} \geq 2^{m/2+3}$ (where the last inequality follows from our assumption on t). Thus the inequality $d(H) \leq \log|T\colon H|$ also holds in this case. ∎

We note that the proof that $r(A_t) \leq t/2$ applies the Classification Theorem. However, an elementary proof of Proposition 1.5 is easily obtained by using Jerrum's bound $r(A_t) \leq t-1$ instead [Jer]. In fact it can be shown that, if T is alternating, then a slightly sharper bound on $d(H)$ holds.

Proof of Theorem 1.4: In view of Theorem 1.3 and the Classification Theorem it suffices to deal with simple groups of Lie type. Given T and H with $|T\colon H| = n$, let $T_0 \leq H$ and c be as in Proposition 3.1. Denote the Lie rank of T by k.

CLAIM: There are $O(k^2)$ choices for T_0 up to conjugacy in T.

Indeed, if $T_0 = 1$ there is nothing to prove, so suppose $T_0 = \widetilde{T_0}$ is quasisimple and naturally embedded in T. It is easy to verify that T has $O(k^2)$ orbits in its action on the subspaces of its natural module. Now, given the natural module U for T_0, there are $O(1)$ choices for the subgroup $T_0 \leq T$. This proves the claim.

Counting the subgroups H up to conjugacy, we may assume that there are $O(k^2)$ choices for the subgroup $T_0 \leq H$. The information on the minimal degrees of permutation representations of groups of Lie type (cf. [KlLi, 5.2.2], [LaSe]) yields immediately $k \leq n$ (this is a very crude bound). Now, given T_0, H will be obtained by adding to T_0 at most $d = \log|H\colon T_0| \leq \log|T\colon H|^{c-1} = (c-1)\log n$ generators $x_1, \ldots, x_d$. Replacing each x_i by $x_i t_i$ ($t_i \in T_0$) will not change the resulting group. Thus, given T_0 there are at most $|T\colon T_0|^{(c-1)\log n} \leq n^{c(c-1)\log n}$ ways to choose H. Putting everything together we see that

$$a_n(T) \leq n \cdot c'n^2 \cdot n^{c(c-1)\log n}.$$

This completes the proof.

Remark: By considering elementary abelian p-subgroups of a group T of Lie type in characteristic p (for fixed p), we easily see that the bounds in Theorem 1.4 and Proposition 1.5 are best possible (apart from the values of the constants).

4. Profinite completions

Let $G = \prod_{n\geq 5} S_n$. Given $n \geq 5$ define $\tau_n, \sigma_n \in S_n$ by $\tau_n = (1,2)$ and $\sigma_n = (1,2,\ldots,n)$. Consider $\tau = (\tau_n)_{n\geq 5}$ and $\sigma = (\sigma_n)_{n\geq 5}$ as elements of G, and let Γ be the abstract subgroup they generate.

For a set J of integers exceeding 4, let Γ_J be the projection of Γ to $G_J = \prod_{n\in J} S_n$. Let Γ_J^0 be the intersection of Γ_J with $A_J = \prod_{n\in J} A_n$. Note that there exists a homomorphism $\psi\colon \Gamma_J\colon \longrightarrow \prod_{n\in J} S_n/A_n$ whose kernel is Γ_J^0 and whose image is an elementary abelian 2-group of rank at most 2. It follows that $|\Gamma_J\colon \Gamma_J^0| \leq 4$, so Γ_J^0 is a finitely generated group.

PROPOSITION 4.1: $\widehat{\Gamma_J^0} \cong \widehat{\mathbb{Z}} \times \prod_{n\in J} A_n$.

Proof: We first establish this for $J = \{n \in \mathbb{Z}\colon n \geq 5\}$.

For $i \geq 5$ let

$$\Delta_i = \langle \tau, \sigma\tau\sigma^{-1}, \ldots, \sigma^{i-2}\tau\sigma^{-(i-2)} \rangle.$$

Set also $\Delta_i^0 = \Delta_i \cap \prod_{n\geq 5} A_n$.

We claim that the groups Δ_i are all finite. Indeed, the projection of Δ_i to S_n is the group generated by $(1,2),(2,3),\ldots,(j-1,j)$ where $j=\min(i,n)$. If $i>n$ these elements generate S_n, but for $n\geq i$ they generate a copy of S_i inside S_n. Furthermore, it is easy to see that the projection of Δ_i into $\prod_{n\geq i} S_n$ is a copy of S_i embedded diagonally in $\prod_{i\geq n} S_n$. This already implies that Δ_i is finite. Since for $n\leq i$, A_n are distinct finite simple groups, we conclude that $\Delta_i^0 \cong \prod_{n=5}^{i} A_n$, embedded in $\prod_{n\geq 5} A_n$ by the map $(x_5,x_6\ldots,x_i)\mapsto(x_5,x_6,\ldots,x_i,x_i,x_i,\ldots)$.

Let $L=\bigcup_{i\geq 5}\Delta_i$. Then L is a locally finite group and by the definition of the Δ_i's we have

$$L\leq \sigma^{-1}L\sigma\leq\sigma^{-2}L\sigma^2\leq\cdots\leq\sigma^{-j}L\sigma^j\leq\cdots.$$

Let $P=\bigcup_{j\geq 0}\sigma^{-j}L\sigma^j$. Then P is also locally finite, and $P\lhd\Gamma$. In fact P is the normal closure of τ in Γ. Thus Γ/P is a cyclic group, generated by the image of σ. Let $K=\widehat{\Gamma}$ be the profinite completion of Γ.

Claim: We have $\overline{P}=\overline{L}$ in K.

It suffices to prove that for every finite quotient F of Γ, the images of P and L in F coincide. Let $\pi\colon\Gamma\longrightarrow F$ be a projection. Then $\pi(L)\leq\pi(\sigma^{-1}L\sigma)$ since $L\leq\sigma^{-1}L\sigma$. But $\pi(\sigma^{-1}L\sigma)=\pi(\sigma)^{-1}\pi(L)\pi(\sigma)$ which has the same order as $\pi(L)$. We conclude that $\pi(L)=\pi(\sigma^{-1}L\sigma)$. It follows in a similar manner that $\pi(L)=\pi(\sigma^{-j}L\sigma^j)$ for all j, so $\pi(L)=\pi(P)$ as required.

Now, set

$$L^0=\bigcup_{i\geq 5}\Delta_i^0,$$

and let

$$P^0=\bigcup_{j\geq 0}\sigma^{-j}L^0\sigma^j.$$

It follows in a similar manner that $\overline{L^0}=\overline{P^0}$ in K.

Since $\Gamma/P\cong\mathbb{Z}$ we have $K/\overline{P}\cong\widehat{\mathbb{Z}}$. Since Γ^0 has finite index in Γ it follows that $\overline{\Gamma^0}\cong\widehat{\Gamma^0}$ and $\widehat{\Gamma^0}/\overline{P^0}\cong\widehat{\mathbb{Z}}$.

Claim: $\overline{L^0}\cong\prod_{n\geq 5}A_n$.

To show this note that, by the definition of L^0, every finite quotient of L^0 is a quotient of Δ_i^0 for some i. By the structure of Δ_i^0 such a quotient is a direct product of alternating groups, each occurring at most once.

Next, L^0 is dense in $\prod_{n\geq 5} A_n$. Therefore, letting $\widehat{\phi}\colon \widehat{\Gamma} \longrightarrow \prod_{n\geq 5} S_n$ be the map induced by the embedding $\phi\colon \Gamma \longrightarrow \prod_{n=5}^{\infty} S_n$, we obtain $\widehat{\phi}(\overline{L^0}) = \prod_{n\geq 5} A_n$. Therefore $\overline{L^0}$ has $\prod_{n\geq 5} A_n$ as a quotient. The information on the composition factors of the finite images of $\overline{L^0}$ enables us to deduce that the kernel is trivial, so

$$\overline{L^0} \cong \prod_{n\geq 5} A_n,$$

where the isomorphism is given by $\widehat{\phi}|_{\overline{L^0}}$. This proves the claim.

Let $\pi = \widehat{\phi}|_{\overline{L}}$. Then π is a monomorphism from $\overline{L}$ to $\prod_{n\geq 5} S_n$, $\pi(\overline{L}) = \widehat{\phi}(\widehat{\Gamma})$ (since $\widehat{\phi}(\sigma) \in \pi(\overline{L})$). We also have $|\pi(\overline{L})\colon \pi(\overline{L^0})| = 4$. Consider the following exact sequence:

$$1 \longrightarrow \overline{L} = \overline{P} \longrightarrow \widehat{\Gamma} \longrightarrow \widehat{\mathbb{Z}} \longrightarrow 1. \tag{1}$$

It is clear that the map $\pi^{-1}\widehat{\phi}\colon \widehat{\Gamma} \longrightarrow \overline{L}$ acts as identity on $\overline{L}$. This shows that $\widehat{\Gamma} \cong \overline{L} \times \widehat{\mathbb{Z}}$.

It follows now that $\widehat{\Gamma^0} = \widehat{\mathbb{Z}} \times \prod_{n\geq 5} A_n$, as asserted.

It remains to prove the proposition for general subsets J. We need the following.

Claim: Let Γ be a discrete group and $\widehat{\Gamma}$ its profinite completion. Let H be a closed normal subgroup of $\widehat{\Gamma}$ with the property that $H \cap \Gamma$ is dense in H. Then the profinite completion of $\Gamma/(H \cap \Gamma)$ is isomorphic to $\widehat{\Gamma}/H$.

The claim follows easily from the universal property of the profinite completion functor. Indeed, given a profinite group G and a homomorphism $\pi\colon \Gamma/(\Gamma \cap H) \longrightarrow G$, we obtain a natural homomorphism $\Gamma \longrightarrow G$ which we denote by ϕ. Now, ϕ can be extended to $\widehat{\phi}\colon \widehat{\Gamma} \longrightarrow G$, and the kernel of $\widehat{\phi}$ contains $\Gamma \cap H$. Since the kernel is closed we see that $\mathrm{Ker}(\widehat{\phi}) \supseteq \overline{\Gamma \cap H} = H$. Therefore $\widehat{\phi}$ factors through $\widehat{\Gamma}/H$, and the claim follows.

To show that $\widehat{\Gamma^0_J} = \widehat{\mathbb{Z}} \times \prod_{n\in J} A_n$ it now suffices to show that $\Gamma^0 \cap \prod_{n\in J^c} A_n$ is dense in $\prod_{n\in J^c} A_n$, where $J^c = \{5,6,7,\dots\} \setminus J$. For this it suffices to show that $\Gamma^0 \cap A_n = A_n$, where A_n is identified here with the nth component of $\prod_{i\geq 5} A_i$. Recall that Δ^0_{n+1} equals the set of elements of the form $(x_5, x_6, \dots, x_n, x_{n+1}, x_{n+1}, \dots)$. Thus $A_n \leq \Delta^0_{n+1} \leq \Gamma^0$, which is what we need.

Proposition 4.1 is proved. ∎

Similar arguments give rise to the following result (based on a group constructed in [LW]).

PROPOSITION 4.2: *Let q be a fixed prime power. Then for every subset $J \subseteq \{n \in \mathbb{Z}: n \geq 2\}$ there exists a finitely generated discrete group $\Gamma(J)$ such that*

$$\widehat{\Gamma(J)} \cong \widehat{\mathbb{Z}} \times \prod_{n\in J} \mathrm{PSL}_n(q).$$

Proof: Given $n \geq 2$, let $\alpha_n, \beta_n \in \mathrm{PSL}_n(q)$ be the images of the matrices

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Choose $\lambda \in \mathbb{F}_q$ such that $\mathbb{F}_q = \mathbb{F}_p[\lambda]$ and let $\gamma_n \in \mathrm{PSL}_n(q)$ be the image of the matrix

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda^{-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Finally, we let $\delta_n \in \mathrm{PSL}_n(q)$ be the image of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ (-1)^{n-1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, let $G = \prod_{n\geq 2} \mathrm{PSL}_n(q)$ and let Γ be the abstract subgroup of G generated by $\alpha = (\alpha_n)$, $\beta = (\beta_n)$, $\gamma = (\gamma_n)$ and $\delta = (\delta_n)$.

Clearly α, β, γ generate a copy of $\mathrm{PSL}_2(q)$ in G, which we denote by Δ_2. For $i > 2$ let

$$\Delta_i = \langle \Delta_2, \delta\Delta_2\delta^{-1}, \ldots, \delta^{i-2}\Delta_2\delta^{-(i-2)} \rangle.$$

Induction on i shows that the projection of Δ_i to $\mathrm{PSL}_n(q)$ is surjective for $n \leq i$, and yields a diagonal copy of $\mathrm{PSL}_i(q)$ for $n \geq i$. As before we deduce that

$$\Delta_i = \{(x_2, x_3, \ldots, x_i, x_i, x_i, \ldots): x_m \in \mathrm{PSL}_m(q) \text{ for all } m\}.$$

In particular, Δ_i is a finite subgroup of Γ.

Let $L = \bigcup_{i\geq 2} \Delta_i$. Then

$$L \leq \delta^{-1} L \delta \leq \cdots \leq \delta^{-j} L \delta^j \leq \cdots$$

and $P = \bigcup_{j\geq 0} \delta^{-j} L \delta^j$ is a locally finite normal subgroup of Γ (which coincides with the normal closure of Δ_2). We see that Γ/P is generated by the image of δ.

As in the proof of Proposition 4.1 we obtain $\overline{L} = \overline{P}$ in $\widehat{\Gamma}$ and

$$\widehat{\Gamma} \cong \widehat{\mathbb{Z}} \times \prod_{n\geq 2} \mathrm{PSL}_n(q).$$

Similarly, given $J \subseteq \{2,3,4,\ldots\}$ we take $H = \Gamma \cap \prod_{n\in J^c} \mathrm{PSL}_n(q)$ and set

$$\Gamma(J) = \Gamma/H.$$

We then have

$$\widehat{\Gamma(J)} = \widehat{\Gamma/H} \cong \widehat{\Gamma}/\prod_{n\in J^c} \mathrm{PSL}_n(q) \cong \widehat{\mathbb{Z}} \times \prod_{n\in J} \mathrm{PSL}_n(q),$$

as required. ∎

5. Proof of main results

We start with a crude result concerning the subgroup growth of a direct product.

LEMMA 5.1: *Let A, B be groups and let n be a positive integer. Then*

(1) $s_n(A\times B) \leq s_n(A)^2 s_n(B)^2 n^{r(B)}$.

(2) $s_n(A\times \mathbb{Z}) \leq n^3 s_n(A)^2$.

Proof: Let $H \leq A\times B$ be a subgroup of index at most n. Let K, L be the projections of H to A, B respectively, and let K_0, L_0 be the intersections of H with A, B respectively. Then $K_0 \lhd K \leq A$ and $|A\colon K_0| \leq n$, and similarly $L_0 \lhd L \leq B$ and $|B\colon L_0| \leq n$. In particular, there are at most $s_n(A)^2 s_n(B)^2$ choices for K, K_0, L, L_0. Given these subgroups, H will be determined once we choose a complement to K/K_0 in $K/K_0 \times L/L_0$, and the number of ways to do that is $|\mathrm{Hom}(L/L_0, K/K_0)|$. Since

$$|\mathrm{Hom}(L/L_0, K/K_0)| \leq |K/K_0|^{d(L/L_0)} \leq n^{r(B)},$$

part 1 follows. Part 2 is a consequence of part 1. ∎

It follows from part 2 of the above lemma that (unless A is almost cyclic) A and $A \times \mathbb{Z}$ (or $A \times \widehat{\mathbb{Z}}$ in the profinite case) have the same growth type.

We denote by $m(G)$ the minimal index of a proper subgroup of G.

LEMMA 5.2: *Let T_i ($i \geq 1$) be pairwise non-isomorphic finite simple groups, and let $f\colon \mathbb{N} \longrightarrow \mathbb{R}$ be a non-decreasing function satisfying $\lim_{n\to\infty} f(n) = \infty$. Then there exists a subset $J \subseteq \mathbb{N}$ such that, if $G = \prod_{j\in J} T_j$, and $j(n)$ is the maximal index $j \in J$ such that $m(T_j) \leq n$, then*

$$a_n(T_{j(n)}) \leq a_n(G) \leq s_n(T_{j(n)})^2 n^{f(n)} \quad \textit{for all } n.$$

Proof: Choose an increasing series $\{j_i\}$ of positive integers such that, for all $k > 1$, we have

$$m(T_{j_k}) \geq s(T_{j_1} \times \cdots \times T_{j_{k-1}})^2 \tag{2}$$

and

$$f(m(T_{j_k})) \geq r(T_{j_1} \times \cdots \times T_{j_{k-1}}) + 1. \tag{3}$$

Let J be the set consisting of the integers j_i, and let G, n, $j(n)$ be as in the lemma. Then all index n subgroups of G contain $\prod_{j\in J, j>j(n)} T_j$, which means that

$$a_n(G) = a_n(T_{j_1} \times \cdots \times T_{j_k}),$$

where $j_k = j(n)$. Set $A = T_{j_k}$ and $B = T_{j_1} \times \cdots \times T_{j_{k-1}}$. Using Lemma 5.1 we see that

$$a_n(G) = a_n(A \times B) \leq s_n(A)^2 s_n(B)^2 n^{r(B)}.$$

By (2) and (3) we have

$$s_n(B)^2 \leq s(B)^2 \leq m(A) \leq n \quad \text{and} \quad r(B) \leq f(m(A)) - 1 \leq f(n) - 1.$$

This yields

$$a_n(G) \leq s_n(A)^2 \cdot n^{f(n)},$$

which proves the upper bound. The lower bound is trivial. ∎

Proof of Theorem 1.1: Applying Lemma 5.2 with $f(n) = \log n/(\log\log n)^2$ we can chose an infinite subset J of integers exceeding 4 such that

$$a_n\left(\prod_{j\in J} A_j\right) \leq s_n(A_{j(n)})^2 n^{\log n/(\log\log n)^2}.$$

Applying Theorem 1.3 we see that

$$s_n(A_{j(n)}) \leq n \cdot n^{c \log n/(\log\log n)^2}.$$

It follows that

$$a_n\left(\prod_{j\in J} A_j\right) \leq n^{C \log n/(\log\log n)^2},$$

for a suitable $C > c$. In view of the lower bound in Lemma 5.2 and the remark at the end of Section 2, it is clear that $\prod_{j\in J} A_j$ has growth type $n^{\log n/(\log\log n)^2}$.

Now, consider the discrete group Γ_J^0 defined in Section 2. The subgroup growth of a group coincides with the subgroup growth of its profinite completion, so applying Proposition 4.1 we obtain

$$a_n(\Gamma_J^0) = a_n(\widehat{\Gamma_J^0}) = a_n\left(\prod_{j\in J} A_j \times \widehat{\mathbb{Z}}\right).$$

By the remark following Lemma 5.1 we see that Γ_J^0 and $\prod_{j\in J} A_j$ have the same growth type.

The proof is complete.

Proof of Theorem 1.2: This follows in a similar manner, by combining Lemma 5.2, Proposition 4.2 and Theorem 1.4.

Proof of Corollary 1.6: Let us show that the abstract groups constructed above have polynomial growth of subnormal subgroups. As usual we may replace our abstract groups by their profinite completions. As in Lemma 5.1 we obtain $a_n^{\triangleleft\triangleleft}(A \times \widehat{\mathbb{Z}}) \leq n^3 a_n^{\triangleleft\triangleleft}(A)^2$. It therefore suffices to prove that, if $G = \prod_{j\in J} T_j$ where J is constructed as in the proof of Lemma 5.2, then $a_n^{\triangleleft\triangleleft}(G) \leq n^c$ for some c.

Since every normal subgroup of G has the form $\prod_{j\in J'} T_j$ for some subset $J' \subseteq J$, it follows by induction that every subnormal subgroup of G has a similar form. Let H be a subnormal subgroup of index n in G. Then $H = \prod_{j\in J'} T_j$ where $J \setminus J'$ is finite. Let j be the maximal element of $J \setminus J'$. Then $|T_j| \leq n$, and condition (2) above readily implies $|T_j| \geq 2^j$. Therefore $j \leq \log n$, so given n there are at most $\log n$ ways to choose j. Once j is given there are at most $2^{j-1} < n$ ways to choose the other elements of $J \setminus J'$, which together determine H. This yields

$$a_n^{\triangleleft\triangleleft}(G) < n \log n.$$

The result follows.

In fact, using more delicate arguments we can prove the following.

PROPOSITION 5.3: *Given any function* $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ *with* $\lim_{n\to\infty} f(n) = \infty$, *there exists an infinite subset* $J \subseteq \{n \in \mathbb{Z}\colon n \geq 5\}$ *such that*

$$a_n^{\triangleleft\triangleleft}(\Gamma_J^0) \leq f(n) \quad \textit{for all } n.$$

Proof: We sketch the argument, leaving the verification of some details to the reader. Since there are no non-trivial homomorphisms from a direct product of nonabelian simple groups to a cyclic group, it follows that every subnormal subgroup of $\widehat{\mathbb{Z}} \times \prod_{j\in J} A_j$ is of the form $K \times L$, where $K \leq \widehat{\mathbb{Z}}$ and $L \triangleleft\triangleleft \prod_{j\in J} A_j$. We see that

$$a_n^{\triangleleft\triangleleft}(\Gamma_J^0) = a_n^{\triangleleft\triangleleft}\left(\widehat{\mathbb{Z}} \times \prod_{j\in J} A_j\right) = \sum_{d|n} a_d^{\triangleleft\triangleleft}\left(\prod_{j\in J} T_j\right).$$

To estimate the right hand side we argue as in the proof of Corollary 1.6. It is easy to verify that, if $J = \{j_k\}_{k\geq 1}$ and j_k grow sufficiently fast (given the function f), then $\sum_{d|n} a_d^{\triangleleft\triangleleft}(\prod_k T_{j_k}) \leq f(n)$. The result follows. ∎

We conclude that, even if we assume that $a_n^{\triangleleft\triangleleft}(G)$ grows extremely slowly (e.g. $a_n^{\triangleleft\triangleleft}(G) \leq \log\log\log n$), still the group G need not be virtually soluble, or linear.

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